

# A SHORT NOTE ON VECTOR BUNDLES ON CURVES

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**ABSTRACT.** In [BL94] Beauville and Laszlo give an interpretation of the affine Grassmannian for  $\mathrm{Gl}_n$  over a field  $k$  as a moduli space of, loosely speaking, vector bundles over a projective curve together with a trivialization over the complement of a fixed closed point. In order to establish this correspondence, they use an abstract descent lemma, which they prove in [BL95]. It turns out, however, that one can avoid this descent lemma by using a simple approximation-argument, which leads to a more direct prove of the above mentioned correspondence.

## 1. INTRODUCTION

There is a well-known correspondence between points of the affine Grassmannian for  $\mathrm{Gl}_n$  and vector bundles on a projective curve together with certain trivializations. Let us recall this correspondence, as Beauville and Laszlo describe it in [BL94].

Let  $X$  be a smooth projective curve over  $k$ ,  $p \in X$  be a closed point, and choose a uniformizer  $z \in \mathcal{O}_{X,p}$ . We fix these data for the rest of these notes. For every  $k$ -algebra  $R$  we set

$$(1.1) \quad \begin{aligned} X_R &:= X \otimes_{\mathrm{Spec} k} \mathrm{Spec} R, & X_R^* &:= \mathrm{Spec}(\mathcal{O}_X(X - \{p\}) \otimes_k R), \\ D_R &:= \mathrm{Spec} R[[z]], & D_R^* &:= R((z)). \end{aligned}$$

These data determine a cartesian diagram of schemes

$$(1.2) \quad \begin{array}{ccc} D_R^* & \xrightarrow{\psi} & X_R^* \\ \downarrow i & & \downarrow j \\ D_R & \xrightarrow{f} & X_R. \end{array}$$

Beauville and Laszlo prove the following

**Proposition 1** ([BL94], Proposition 1.4). *The functor*

$$\mathrm{L} \mathrm{Gl}_n : R \mapsto \mathrm{Gl}_n(R((z)))$$

*on the category of  $k$ -algebras is isomorphic to the functor which associates to  $R$  the set of isomorphism classes of triples  $(E, \rho, \sigma)$ , where  $E$  is a vector bundle of rank  $n$  over  $X_R$ , and  $\rho$  and  $\sigma$  are trivializations of  $E$  over  $X_R^*$  and  $D_R$ , respectively.*

As a consequence they obtain

**Proposition 2** ([BL94], Proposition 2.1 and Remark 2.2). *The affine Grassmannian for  $\mathrm{Gl}_n$ , which is by definition the fpqc-sheafification of the functor  $R \mapsto \mathrm{Gl}_n(R((z))) / \mathrm{Gl}_n(R[[z]])$ , is isomorphic to the functor which associates to  $R$  the set*

of isomorphism classes of pairs  $(E, \rho)$ , where  $E$  is a vector bundle of rank  $n$  over  $X_R$ , and  $\rho$  is a trivialization of  $E$  over  $X_R^*$ .

The interesting part in the proof of Proposition 1 is to see why the data of trivial vector bundles of rank  $n$  on  $D_R$  and  $X_R^*$ , respectively, together with a transition function over  $X_R^*$ , determine a vector bundle on  $X_R$ . This is not a classical descent situation, since if  $R$  is not Noetherian,  $D_R$  is in general not flat over  $X_R$ . In [BL95] Beauville and Laszlo prove that descent holds nonetheless.

In the present notes we present an alternative proof of Proposition 1 using the following strategy. We define the subring  $A_R \subset R[[z]]$  as a certain localization of  $\mathcal{O}_{X,p} \otimes_k R$ , which depends functorially on  $R$  and determines a flat neighborhood of the locus  $z = 0$  in  $X_R$ . Let us write  $\Delta_R = \text{Spec } A_R$  and  $\Delta_R^* = \text{Spec } A_R[1/z]$ . Then  $\Delta_R \amalg X_R^* \rightarrow X_R$  is an fppf-covering, and if we could replace  $D_R$  by  $\Delta_R$  and  $D_R^*$  by  $\Delta_R^*$  in the formulation of Proposition 1, then this proposition would immediately follow by faithfully flat descent. Indeed, we will show below how to arrive at this situation using a simple approximation argument. Moreover, the concrete situation will turn out to be not only fppf-local, but even Zariski-local, so that descent of vector bundles holds trivially.

## 2. VECTOR BUNDLES ON A SMOOTH CURVE

Note that the choice of a uniformizer  $z \in \mathcal{O}_{X,p}$  determines an inclusion  $(R \otimes_k \mathcal{O}_{X,p}) \subset R[[z]]$ ,  $R[[z]]$  being the completion with respect to the  $z$ -adic valuation. For each  $f \in (R \otimes_k \mathcal{O}_{X,p}) \cap R[[z]]^\times$  we define  $S_{R,f} := (R \otimes_k \mathcal{O}_{X,p})_f \subset R[[z]]$ . The union of all these rings, for varying  $f$ , will be denoted  $A_R$ . Writing  $\Delta_R := \text{Spec } A_R$  and  $\Delta_R^* := \text{Spec } A_R[1/z]$  we have a cartesian diagram

$$\begin{array}{ccc} \Delta_R^* & \xrightarrow{\psi} & X_R^* \\ \downarrow \iota & & \downarrow j \\ \Delta_R & \xrightarrow{\varphi} & X_R. \end{array}$$

Moreover we set  $U_{R,f} := \text{Spec } S_{R,f}$ .

**Lemma 3.** *The morphism  $D_R \amalg X_R^* \rightarrow X_R$  is surjective. Thus  $\Delta_R \amalg X_R^* \rightarrow X_R$  is an fppf-, and  $U_{R,f} \amalg X_R^* \rightarrow X_R$  is a Zariski-covering for each  $f \in (R \otimes_k \mathcal{O}_{X,p}) \cap R[[z]]^\times$ .*

*Proof.* Let  $P$  be a point of  $X_R$  and let  $A = (\mathcal{O}_X \otimes R)_P$  be the local ring at  $P$ . Either  $z$  is invertible in  $A$  – then  $P \in X_R^*$  – or  $z$  is in the maximal ideal  $\mathfrak{p} \subset A$ . In the latter case we consider  $\text{can} : A \rightarrow \hat{A} = \varprojlim A/z^N$  and the ideal  $\hat{\mathfrak{p}} = \varprojlim \mathfrak{p}/z^N$ . Passing to the inverse limit over the short exact sequences

$$0 \rightarrow \mathfrak{p}/(z^N) \rightarrow A/(z^N) \rightarrow A/\mathfrak{p} \rightarrow 0$$

we obtain  $\text{can}^{-1}(\hat{\mathfrak{p}}) = \mathfrak{p}$ , and the commutative square

$$\begin{array}{ccc} \text{Spec } \hat{A} & \longrightarrow & \text{Spec } R[[z]] = D_R \\ \downarrow & & \downarrow \\ \text{Spec } A & \longrightarrow & X_R. \end{array}$$

shows that  $\hat{\mathfrak{p}} \cap R[[z]] \subset R[[z]]$  is a preimage of  $P$  in  $D_R$ . □

Let  $T$  be the functor on the category of  $k$ -algebras, which associates to a  $k$ -algebra  $R$  the set of isomorphism classes of triples  $(E, \rho, \sigma)$ , where  $E$  is a vector bundle of rank  $n$  on  $X_R$ , and

$$\begin{aligned}\rho &: \mathcal{O}_{X_R^*}^n \xrightarrow{\cong} E|_{X_R^*}, \\ \sigma &: \mathcal{O}_{\Delta_R}^n \xrightarrow{\cong} E|_{\Delta_R}\end{aligned}$$

are trivializations. To each isomorphism class  $[(E, \rho, \sigma)] \in T(R)$  we may assign the respective ‘transition matrix over  $\Delta_R^*$ ’. This is independent of the actual representative of  $[(E, \rho, \sigma)]$  and hence determines a morphism of functors

$$\Phi(R) : T(R) \rightarrow \mathrm{Gl}_n(A_R[1/z]); \quad (E, \rho, \sigma) \mapsto \Gamma(X_R, (\rho|_{\Delta_R^*}) \circ (\sigma^{-1}|_{\Delta_R^*})).$$

**Proposition 4.** *The morphism  $\Phi(R)$  defined above is an isomorphism of functors.*

*Proof.* We have to construct an inverse for  $\Phi(R)$ . To this end, we choose a matrix  $g \in \mathrm{Gl}_n(A_R[1/z])$  and consider the following diagram of quasi-coherent sheaves on  $X_R$ ,

$$\begin{array}{ccc} E & \xrightarrow{\quad} & \mathcal{O}_{X_R^*}^n \\ \downarrow & & \downarrow \text{can} \\ \mathcal{O}_{\Delta_R}^n & \xrightarrow{\text{can}} \mathcal{O}_{\Delta_R^*}^n \xrightarrow{g} & \mathcal{O}_{\Delta_R^*}^n \end{array}$$

where  $E$  is uniquely determined up to isomorphism by requiring that the diagram be cartesian. (By abuse of notation we do not indicate the obvious push-forwards to  $X_R$  in this diagram.) It is easy to check (by pullback to  $\Delta_R$  and  $X_R^*$ , respectively) that this diagram determines trivializations of  $E$  over  $\Delta_R$  and  $X_R^*$ . The transition function for these two trivializations is equal to  $g$  by construction.

To see that this construction indeed gives an inverse for  $\Phi(R)$  it remains to check that  $E$  is a vector bundle. This is immediate by Lemma 3 together with faithfully flat descent, or by the following elementary argument: the matrix  $g$  involves only finitely many elements of  $A_R[1/z]$ , whence in fact  $g \in S_{R,f}[1/z]$  for some  $f \in (R \otimes_k \mathcal{O}_{X,p}) \cap R[[z]]^\times$ . This shows that  $E$  can as well be obtained by gluing trivial bundles over  $U_{R,f}$  and over  $X_R^*$ , respectively. Now, since  $U_{R,f} \subset X_R$  is Zariski-open, this shows that  $E$  is a vector bundle.  $\square$

### 3. ‘FORMAL’ DESCENT OF VECTOR BUNDLES

Let us now consider the situation introduced at the beginning in diagram (1.2), where we consider the formal neighborhood  $D_R = \mathrm{Spec} R[[z]]$  of  $\mathrm{Spec} R \times \{p\} \subset X_R$ .

By  $\hat{T}$  we denote the functor, which associates to every  $k$ -algebra  $R$  the set of isomorphism classes of triples  $(E, \rho, \sigma)$ , where  $E$  is a vector bundle of rank  $n$  over  $X_R$  and

$$\begin{aligned}\rho &: \mathcal{O}_{X_R^*}^n \xrightarrow{\cong} E|_{X_R^*}, \\ \sigma &: \mathcal{O}_{D_R}^n \xrightarrow{\cong} E|_{D_R}\end{aligned}$$

are trivializations.

As in the previous section, we obtain a functorial morphism  $\hat{\Phi}(R) : \hat{T}(R) \rightarrow \mathrm{Gl}_n(R((z)))$  by assigning to each triple  $(E, \rho, \sigma)$  the corresponding transition function over  $D_R^*$ .

**Theorem 5** ([BL94], Proposition 1.4). *The morphism  $\hat{\Phi}$  is an isomorphism of functors.*

*Proof.* In order to construct an inverse for  $\hat{\Phi}$ , i.e. to construct a triple  $(E, \rho, \sigma)$  from a given  $\gamma \in \mathrm{Gl}_n(R((z)))$ , we proceed exactly as in the proof of Proposition 4. The only non-trivial thing to check is that the quasi-coherent sheaf  $E$ , defined so to make the diagram

$$(3.1) \quad \begin{array}{ccc} E & \xrightarrow{\quad} & \mathcal{O}_{X_R}^n \\ \downarrow & & \downarrow \text{can} \\ \mathcal{O}_{D_R}^n & \xrightarrow{\text{can}} \mathcal{O}_{D_R^*}^n \xrightarrow{\gamma} & \mathcal{O}_{D_R^*}^n \end{array}$$

cartesian, is a vector bundle over  $X_R$ . We do this by reducing to a situation where Proposition 4 applies. More precisely, Lemma 6 below shows that every  $\gamma \in \mathrm{Gl}_n(R((z)))$  can be written as a product  $\gamma = g \cdot \delta$ , where  $g \in \mathrm{Gl}_n(A_R[1/z])$  and  $\delta \in \mathrm{Gl}_n(R[[z]])$ .

Thus diagram (3.1) ‘decomposes’ likewise, and yields the big diagram

$$\begin{array}{ccccccc} E & \xlongequal{\quad} & E & \xrightarrow{\quad} & \mathcal{O}_{X_R}^n & & \\ \downarrow & & \downarrow & & \downarrow \text{can} & & \\ & & \mathcal{O}_{\Delta_R}^n & \xrightarrow{\text{can}} & \mathcal{O}_{\Delta_R^*}^n & \xrightarrow{g} & \mathcal{O}_{\Delta_R^*}^n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_{D_R}^n & \xrightarrow[\delta]{\simeq} & \mathcal{O}_{D_R}^n & \xrightarrow{\text{can}} & \mathcal{O}_{D_R^*}^n & \xrightarrow{g} & \mathcal{O}_{D_R^*}^n \end{array}$$

The two small squares in this diagram are trivially cartesian, while the big rectangle coincides with the square (3.1), and is thus cartesian by definition of  $E$ . Consequently, the upper rectangle is cartesian, which proves that  $E$  is nothing but the vector bundle corresponding to the transition matrix  $g \in \mathrm{Gl}_n(A_R[1/z])$  under the correspondence of Proposition 4.  $\square$

**Lemma 6.** *We have  $\mathrm{Gl}_n(R((z))) = \mathrm{Gl}_n(A_R[1/z]) \cdot \mathrm{Gl}_n(R[[z]])$ .*

*Proof.* We set  $B := \cup_{P \in R[z] \cap R[[z]]^\times} R[z, z^{-1}, P^{-1}] \subset R((z))$  (Note that the ring  $B \cap R[[z]]$  is equal to the ring  $A_R$  in the case  $X = \mathbb{P}_k^1$ ). Since  $B \subset A_R[1/z]$ , it suffices to check that  $\mathrm{Gl}_n(R((z))) = \mathrm{Gl}_n(B) \cdot \mathrm{Gl}_n(R[[z]])$ . First we note that  $\mathrm{Gl}_n(R[[z]]) \subset \mathrm{Gl}_n(R((z)))$  is open: Namely,  $\det : \mathrm{Mat}_n(R[[z]]) \rightarrow R[[z]]$  is continuous and  $R$  carries the discrete topology, and thus  $R^\times \subset R$  is open. This shows that  $\mathrm{Gl}_n(R[[z]]) \subset \mathrm{Mat}_n(R[[z]]) \subset \mathrm{Mat}_n(R((z)))$  are two open inclusions, so  $\mathrm{Gl}_n(R[[z]]) \subset \mathrm{Gl}_n(R((z)))$  is as well open. As a second step we deduce from Lemma 7 below that  $\mathrm{Gl}_n(B) = \mathrm{Gl}_n(R((z))) \cap \mathrm{Mat}_n(B)$ . Since  $\mathrm{Mat}_n(B) \subset \mathrm{Mat}_n(R((z)))$  is dense and  $\mathrm{Gl}_n(R((z))) \subset \mathrm{Mat}_n(R((z)))$  is open, we conclude that  $\mathrm{Gl}_n(B) \subset \mathrm{Gl}_n(R((z)))$  is dense.

These two statements together imply that  $\mathrm{Gl}_n(B) \cdot \mathrm{Gl}_n(R[[z]])$  is dense and closed in  $\mathrm{Gl}_n(R((z)))$ , whence the lemma.  $\square$

**Lemma 7.** *The subring  $B \subset R((z))$  defined above satisfies  $B^\times = R((z))^\times \cap B$ .*

*Proof.* We consider  $f \in R((z))^\times \cap B$ . By multiplying with a suitable  $P \in R[z] \cap R[[z]]^\times$ , we may reduce to the case  $f \in R((z))^\times \cap R[z, z^{-1}]$ . Such an  $f$  has the form  $f = -N + Q$ , where  $N \in R[z, z^{-1}]$  is a nilpotent Laurent polynomial and the leading coefficient of  $Q \in R((z))^\times$  is a unit in  $R$ . Using the formula  $(-N + Q)(N^i + N^{i-1}Q + \cdots + Q^i) = (-N^i + Q^i)$  we may assume that  $f = Q^i$ , i.e. has a leading coefficient in  $R^\times$ . Multiplying with  $z^m$  for a suitable  $m \in \mathbb{Z}$  we obtain  $z^m f \in R[z] \cap R[[z]]^\times$ , which is invertible in  $B$  by construction.  $\square$

The property of the ring  $B$  which is exhibited in the last lemma is crucial for our strategy of approximation to work. This is what forces us to consider the, at first glance, rather artificial rings  $A_R$  instead of for example just  $\mathcal{O}_{X,p} \otimes R$ . The latter would not contain the ring  $B$ , and in particular would not have the property of Lemma 7.

## REFERENCES

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